2010 22nd International Conference on Tools with Artificial Intelligence

An Algorithm For The Least Square-Fitting of Ellipses

Nicola Greggio[§†], Alexandre Bernardino[†], Cecilia Laschi[§], Paolo Dario[§‡] and José Santos-Victor[†]
[§] *ARTS Lab - Scuola Superiore S.Anna, Polo S.Anna Valdera*
Viale R. Piaggio, 34 - 56025 Pontedera, Italy
[‡] *CRIM Lab - Scuola Superiore S.Anna, Polo S.Anna Valdera*
Viale R. Piaggio, 34 - 56025 Pontedera, Italy
[†] *Instituto de Sistemas e Robótica, Instituto Superior Técnico*
1049-001 Lisboa, Portugal

Email: n.greggio@isr.ist.utl.pt

***Abstract*—In this paper we propose a new algorithm for the least square fitting of ellipses from scattered data. Originally based on the one proposed by Fitzgibbon *et Al* in 1999, our procedure is able to overcome the numerical instability of that algorithm. We test our approach versus the latter and another approach with different ellipses. Then, we present and discuss our results.**

***Keywords*-Computer Vision, Constraint Programming, Least Squares Methods, Conditioning and Ill-conditioning, Real-Time Systems**

I. Introduction

The ellipse is the one of the most common pattern to be recognized, since it occurs within several different contexts. Ellipses are present in many practical situations, such as in astronomy [1], medicine [2] , and robotics [3] [4].

Two main approaches are used for ellipse pattern recognition: Clustering/voting (CV) - Hough Transform (HT) [5], the moment method [6], Kalman filtering [7], RANSAC [8], fuzzy clustering [9] - and Least Square (LS) techniques - performed with geometrical [10] or algebraic distance minimization [11] [12].

The algorithm of Fitzgibbon *et Al.* [11] surely represents an important achievement in algebraic fitting, being the first approach solving the problem with a unique solution in a closed form, i.e. even thought without using iterative (and therefore not real-time and time consuming) methods. However, in recent years some limitations came out, together with the relative improvements. The original solution in [11] may lead to an erratic or impossible solution of the fitting optimal solution when the data points lie exactly on the ideal ellipse curve. In 2006 Maini [12] proposed an affine transformation for solving the ill-conditioning of the scatter matrix. He also proposed a resampling procedure that perturbs the data points with gaussian noise in the case of they are too close to the ellipse. However, since this has to be applied M times, eventually averaging the results, it takes a consistent computational burden.

In this paper we propose a solution for the instability of the original Fitzgibbon algorithm [11].

The paper is organized as follows: First, in sec. II we introduce our algorithm. Then, in sec. II-A we describe our approach for solving the numerical instability still present in [11]. We propose our experiments for validating our work, complete with the discussion of the results in sec. III. Finally in sec. IV we conclude.

II. LCSE

In this section we present the solution of the instability of the original algorithm due to the lying of the data points to the ideal ellipse curve, which greatly improves those suggested in [12].

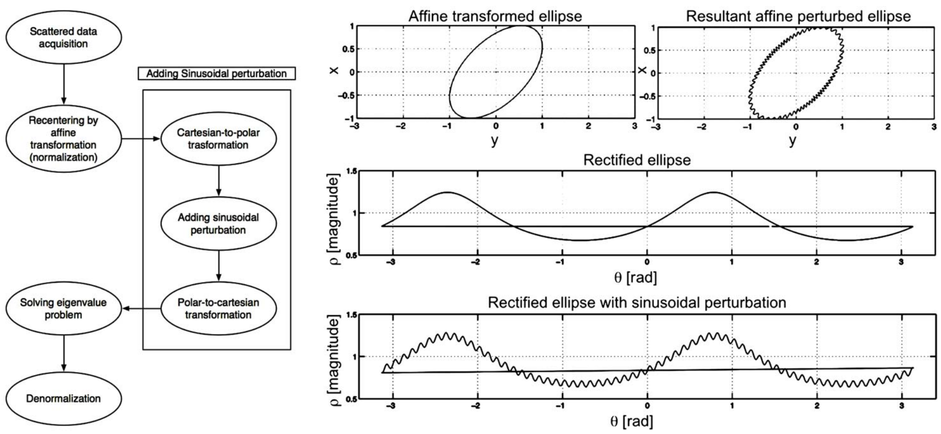
Since now, we will refer to the following abbreviations:

- B2AC: [11];
- EDFE: [12];
- LCSE: our new method.

A. Instability of the exact ellipse solution

1) ***Perturbing function****:* Our idea is not to apply a random noise (even though with a precise probability density function, e.g. gaussian, as in [12]) to the ellipse data, but a deterministic disturb. It can be argued that, due to its formulation based on the minimization least square algebraic distance between data points and the ideal curve representation, this kind of algorithms is not robust to occlusion. Consequently, this approach cannot discriminate the *bad* data from the original ones, and the final result will be affected by this necessarily. Therefore, the perturbation must be have zero average. By seeing the ellipse as a linear function of the angle θ, it is possible to observe that any symmetric periodic function with period taken as integer multiplier of 1 added to the original scattered data leaves the ellipse average unaltered in polar coordinates. In our case we add a sinusoidal function. It can be suggested that other functions can perform the same task as well, such as triangle wave or square wave. However, we choose the sinusoidal wave for its implementation simplicity.

1082-3409/10 $26.00 © 2010 IEEE
DOI 10.1109/ICTAI.2010.124



(a) (b)

Figure 1. (a) The LCSE whole scheme. Here, the perturbing procedure is highlighted; (b) Sinusoidal perturbation. The first two top pictures show the original ellipse and the perturbed one in cartesian coordinates. The third and fourth plot sow the original and the perturbed ellipse in polar coordinates, instead.

*2) **The overall procedure**:* The procedure, showed in Fig. 19(a), is as follows:

- **(a)** First of all, we normalize the ellipse points by applying the affine transformation of [12] (see the left down plot - *Affine transformed ellipse*).
- **(b)** Therefore, we pass from the cartesian coordinates (x, y) to the polar ones (ρ, θ). In polar coordinates the ellipse results as in the first plot of Fig. 1(b) - *Rectified ellipse*.
- **(c)** Addition of the perturbing function. The polar transformed ellipse with the sinusoidal perturbation is represented in the second plot of Fig. 1(b) - *Rectified ellipse with sinusoidal perturbation*. This results, for the point i:

$$\hat{\rho_i} = \rho_i + A \cdot sin(2\pi f \theta_i); \tag{1}$$

- **(d)** Therefore, once the ellipse is remapped in cartesian coordinates, its results equally slightly perturbed inside and outside its ideal curve. It results, for the point i:

$$\begin{aligned} \hat{x_i} &= \hat{\rho_i} \cdot \cos(\theta_i); \\ \hat{y_i} &= \hat{\rho_i} \cdot \sin(\theta_i); \end{aligned} \tag{2}$$

  This is shown in the right down plot of Fig. 1(b).
- **(e)** Now the ellipse is ready to be fitted.
- **(f)** Finally, the affine anti-transformation has to be applied [12].

III. Experimental Validation

We tested the numerical instability of the original algorithm [11], comparing it versus the opproach in [12], and versus our technique. Since the numerical instability occur on perfect ellipses, we tried these algorithms on these.

We will perform two kinds of experiments:

A. The first one is aimed to find the algorithm that is capable of recognize the ellipse (intended as not crashing due to the trivial solution $\mathbf{a}_6 = \mathbf{0}_6$) with the lowest noise amplitude (NA). Since inserting a perturbation, gaussian or sinusoidal, is invasive, i.e. it always alters the goodness of the original data, our aim is to to establish which algorithm, EDFE or LCSE, gives rise to the best percentage of recognition success with the minimum disturb (lowest NA) to the original ellipse.
B. The second one is aimed to determine which approach gives rise to the best percentage in ellipse recognition as function of the root mean square error (RMSE).

So far, we generated different ellipses by varying their center coordinates, their rotation angle, their major and minor axes, and the number of points they are composed by.

- A: Fig. 2(a) shows the results of the trials. The curves describing our algorithm's results are in red. The EDFE is shown with different repetitions M's values and different colors. Then the B2AC is reported in blue.

- B: Fig. 2(b) and 2(c) show the percentage recognition success as function of the RMSE for the EDFE algorithm, with different values for M, and for our technique. Due to our approach's better performance, the two plots have been separated due to the different RMSE scale. Clearly, the technique requiring the lowest NA for reaching the 100% fitting success would be that providing the minor RMSE for achieving the same results, due to the lower perturbation introduced. In Fig. 2(b) we tested EDFE with the same values for M used in the previous section.

- NA amplitude and frequency choice: We performed several tests with different ellipses by varying both the amplitude and the frequency of the sinusoid, in order to find the ones that minimize the RMSE. In our findings it emerges that by increasing the sinusoid amplitude the error increases (as expected). However, by increasing the frequency it remains the same, for all the amplitudes. Therefore, we can assert

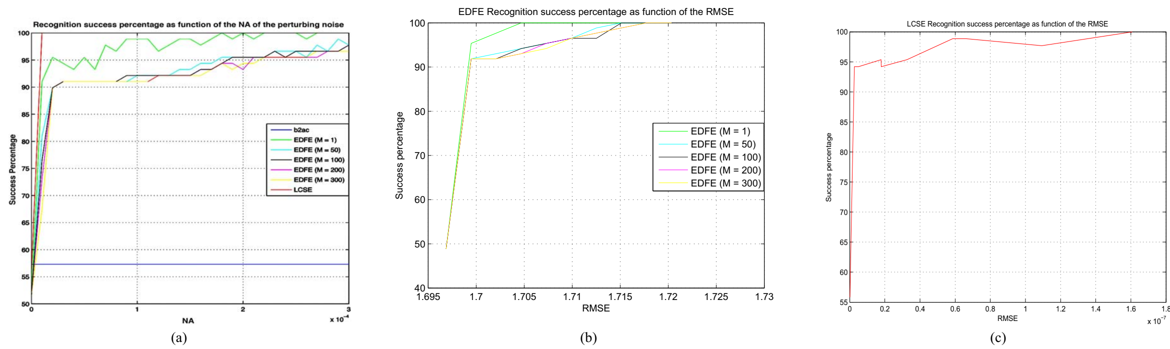


Figure 2. (a) Percentage success in recognizing the ellipse. This shows the success rate in overcoming the original Fitzgibbon numerical instability for our algorithm (LCSE), in red, EDFE (in various colors, depending on the number of resampling procedure M applied, and those of the Fitzgibbon algorithm (B2AC, in blue); (b) LCSE percentage success in recognizing the ellipse as function of the RMSE, (c) EDFE percentage success in recognizing the ellipse as function of the RMSE for different values of the number of resampling procedure repetitions M.

that the frequency does not affect the sinusoidal perturbation in our procedure, in terms of final RMSE.

- Limitations of the proposed approach: Although this technique leads to a good performance in practice, singularities and bias are not always avoided. Rather, their occurrence is more rare. For instance, the input points can be biased when they lie exactly on the peaks of the deterministic noise, or singularities will occur when they lie exactly where the noise is null. Nevertheless, in the majority of cases the instability of the original approach has been sidestepped.

IV. Conclusion

In this paper we proposed a new algorithm for the least square fitting of ellipses from scattered data. This is able to overcome the numerical instability of the original algorithm, proposed by Fitzgibbon *et Al* in 1999 [11]. Over the past years some improvements have been made, e. g. that presented in [12]. We showed that our approach can solve the original numerical instability better than the latter technique, both in terms of computational requirements and RMSE. Finally, we presented and discussed the validity of our results.

Acknowledgements

This work was supported by the European Commission, Project IST-004370 RobotCub and FP7-231640 Handle, and by the Portuguese Government - Fundação para a Ciência e Tecnologia (ISR/IST pluriannual funding) through the PIDDAC program funds and through project BIO-LOOK, PTDC / EEA-ACR / 71032 / 2006.

References

[1] C. F. Gauss, “Theory of the motion of the heavenly bodies moving about the sun in conic sections (theoria motus corporum coelestium in sectionibus conicis solem ambientum),” *Translation by C.H. Davis, Dover, New York*, vol. (First published in 1809), 1963.

[2] K. Paton, “Conic sections in chromosome analysis,” *Pattern Recognition*, vol. 2, pp. 39–51, 1970.

[3] N. Greggio, L. Manfredi, C. Laschi, P. Dario, and M. Carrozza, “Robotcub implementation of real-time least-square fitting of ellipses.” *IEEE-RAS International Conference on Humanoid Robots (HUMANOIDS-08) - Daejeon (Korea) December 1-3*, 2008.

[4] M. Vincze, “Robust tracking of ellipses at frame rate,” *Pattern Recognition*, vol. 34, pp. 487–498, 2001.

[5] R. K. K. Yin, P. K. S. Tam, and N. K. Leung, “Modification of hough transform for circles and ellipses detection using 2-d array,” *Pattern Recognition*, vol. 25, no. 9, pp. 1007–1022, 1992.

[6] H. S. K. Voss, “Invariant fitting of planar objects by primitives,” *IEEE Trans. PAMI*, vol. 19, no. 1, pp. 80–84, 1997.

[7] J. Porrill, “Fitting ellipses and predicting confidence envelopes using a bias corrected kalman filter,” *Image and Vision Computing*, vol. 8, pp. 37–41, 1990.

[8] M. Vincze, M. Ayromlou, and M. Zillich, “Fast tracking of ellipses using edge-projected integration of cues,” *Pattern Recognition. Proceedings. 15th International Conference on*, vol. 4, no. ISBN: 0-7695-0750-6, pp. 72–75, 2000.

[9] F. Dufrenois, “Ellipse fitting with uncertainty and fuzzy decision stage for detection. application in videomicroscopy.” *Lecture Notes in Computer Science*, vol. 2143, no. ISSN0302-9743 (Print) 1611-3349 (Online), pp. 432–443, 2001.

[10] M. Harker and P. O’Leary, “First order geometric distance (the myth of sampsonus).” *British Machine Vision Conf.*, pp. 87–96, 2006.

[11] A. Fitzgibbon, M. Pilu, and R. Fisher, “Direct least square fitting of ellipses,” *IEEE Trans. PAMI*, vol. 21, pp. 476–480, 1999.

[12] E. S. Maini, “Enhanced direct least square fitting of ellipses,” *IJPRAI*, vol. 20, no. 6, pp. 939–954, 2006.